Unitary group subjoinings

1980 J. Phys. A: Math. Gen. 13 3563

(http://iopscience.iop.org/0305-4470/13/12/007)

J. Phys. A: Math. Gen. 13 (1980) 3563–3583. Printed in Great Britain

Unitary group subjoinings

R C King

Mathematics Department, University of Southampton, Southampton S09 5NH, UK

Received 17 June 1980

Abstract. The subjoining of one compact Lie group H to another such group G is discussed with particular reference to the cases for which G = U(N) and H = U(n). It is shown that maximal subjoinings of these unitary groups are specified by means of the monomial symmetric fu .ctions. Subjoinings, which are defined in terms of mappings between weight spaces, are studied through the properties of characters of the irreducible representations. The branching rules corrresponding to subjoinings are found to involve plethysms. Methods of evaluating the appropriate plethysms are illustrated, some of which make use of subjoining chains whilst others exploit the Weyl symmetry groups of G and H to obtain results more directly. The fact that maximal embeddings are special cases of non-maximal subjoinings is demonstrated and discussed.

1. Introduction

Recently a relation between semi-simple Lie algebras has been introduced by Patera and Sharp (1980) which includes the embedding of one such algebra in another as a special case. This generalisation of embedding has been called subjoining.

The definition of subjoining depends upon the existence of a branching rule projecting the set of weights of each finite-dimensional irreducible representation of one semi-simple Lie algebra onto a set of weights associated with another semi-simple Lie algebra. This latter set is, in general, the difference between two sets, each of which is a set of weights of a collection of finite-dimensional irreducible representations.

Patera *et al* (1980) have given several examples of subjoinings and a start has been made on the problem of enumerating all maximal subjoinings. It is shown here that some of these can be easily understood in terms of the properties of symmetric functions. In the case of the unitary groups these functions enter the discussion quite naturally since the characters of the irreducible representations of the unitary groups are the symmetric functions known as Schür functions, as pointed out by Littlewood (1950). Alternative bases for the set of all symmetric functions exist and, after a general discussion of the nature of subjoinings and their specification in § 2, the power-sum symmetric functions are shown in § 3 to specify maximal subjoinings of the unitary group to itself.

In § 4 this idea is generalised through the use of multiplicative power-sum symmetric functions. However, by virtue of their multiplicative nature, this leads to non-maximal subjoinings.

Following a discussion of the role played by the Weyl symmetry group in relating equivalent subjoinings, it is shown in § 5 that the monomial symmetric functions serve

0305-4470/80/123563+21$01.50

to specify maximal subjoinings. Furthermore the maximal embeddings defined by Schür functions are discussed in § 6 where it is demonstrated that they are, in general, non-maximal subjoinings.

In each of §§ 3, 4, 5 and 6 branching rules are calculated using plethysms of S functions introduced by Littlewood (1950) and tabulated by, for example, Ibrahim (1950), Butler and Wybourne (1971) and Vanagas (1971). The fact that these plethysms are themselves defined by mappings between weight spaces has been exploited elsewhere (King and Plunkett 1976) and this point is taken up in § 7 in order to demonstrate the ease with which branchings associated with subjoinings may be calculated.

Finally, in the concluding § 8, an important step is taken towards the goal of enumerating all maximal subjoinings.

2. The specification of subjoinings

Following Patera and Sharp (1980, § 6), if g and h are semi-simple Lie algebras of rank k_{g} and k_{h} respectively, with $k_{\mathrm{g}} \geqslant k_{\mathrm{h}}$, then h is said to be subjoined to g, signified by writing g>h, if and only if there exists a real, $k_{\mathrm{h}} \times k_{\mathrm{g}}$, matrix $\mathscr{P}$ such that for all finite-dimensional, irreducible representations $\boldsymbol{\lambda}$ of g

$$\mathscr{P} W^{\lambda} = \sum_{\mu} B^{\lambda}_{\mu} W^{\mu} \tag{2.1}$$

where W^{λ} is the set of weights of $\boldsymbol{\lambda}$, W^{μ} is the set of weights of $\boldsymbol{\mu}$, and the summation is carried out over all finite-dimensional irreducible representations $\boldsymbol{\mu}$ of h. The branching multiplicity coefficients B^{λ}_{μ} are integers: positive, zero or negative.

Since this definition, (2.1), only involves the weights of finite-dimensional representations of g and h it also serves to define the subjoining of H to G again signified by writing G > H, where G and H are the compact Lie groups associated with g and h respectively. It is then possible to redefine the concept of subjoining in terms of the characters of finite-dimensional representations of compact Lie groups.

The character $\chi^{\lambda_{\mathrm{G}}}_{\psi}$ of a finite-dimensional, unitary, irreducible representation $\boldsymbol{\lambda}_{\mathrm{G}}$ of the compact Lie group G associated with a Lie algebra g of rank k_{g} may be written in the form

$$\chi^{\lambda_{\mathrm{G}}}_{\psi} = \sum_{m} M^{\lambda_{\mathrm{G}}}_{m} \exp(\mathrm{i}\boldsymbol{m} \cdot \boldsymbol{\psi}) \tag{2.2}$$

where $\boldsymbol{\psi} = (\psi_1, \psi_2, \ldots, \psi_{d_{\mathrm{G}}})$ is a set of real class parameters subject to $(d_{\mathrm{G}} - k_{\mathrm{g}})$ constraints (King and Al-Qubanchi 1980). This expansion serves to define the weight vectors $\boldsymbol{m} = (m_1, m_2, \ldots, m_{d_{\mathrm{G}}})$ and their multiplicities $M^{\lambda_{\mathrm{G}}}_{m}$ in the irreducible representation $\boldsymbol{\lambda}_{\mathrm{G}}$. A similar expression may be written down for the characters $\chi^{\mu_{\mathrm{H}}}_{\phi}$ of the irreducible representation $\boldsymbol{\mu}_{\mathrm{H}}$ of the compact Lie group H, where $\boldsymbol{\phi} = (\phi_1, \phi_2, \ldots, \phi_{d_{\mathrm{H}}})$ is a set of real class parameters subject to $(d_{\mathrm{H}} - k_{\mathrm{h}})$ constraints if k_{h} is the rank of the corresponding Lie algebra h.

With this notation the analogue of (2.1) is the statement that the compact Lie group H is subjoined to the compact Lie group G if and only if there exists a linear operator Q which maps the vector of class parameters $\boldsymbol{\phi}$ of H to a vector $\boldsymbol{\psi} = Q\boldsymbol{\phi}$ of class parameters of G such that for all finite-dimensional irreducible representations $\boldsymbol{\lambda}_G$ of G

$$\chi_{\boldsymbol{\psi}}^{\boldsymbol{\lambda}_G} = \chi_{Q\boldsymbol{\phi}}^{\boldsymbol{\lambda}_G} = \sum_{\boldsymbol{\mu}_H} B_{\boldsymbol{\mu}_H}^{\boldsymbol{\lambda}_G} \chi_{\boldsymbol{\phi}}^{\boldsymbol{\mu}_H} \tag{2.3}$$

where the summation is carried out over all finite-dimensional irreducible representations $\boldsymbol{\mu}_H$ of H and the branching multiplicities $B_{\boldsymbol{\mu}_H}^{\boldsymbol{\lambda}_G}$ are, as before, integers: positive, zero or negative.

The self-dual nature of the Euclidean spaces of dimension d_G and d_H used to specify the class parameter vectors $\boldsymbol{\psi}$ and $\boldsymbol{\phi}$, and the corresponding weight vectors, ensures that corresponding to Q there exists a weight projection operator P given by

$$P = Q^{\mathrm{T}} \tag{2.4}$$

which is such that

$$\exp(\mathrm{i}\boldsymbol{m} \cdot Q\boldsymbol{\phi}) = \exp(\mathrm{i}P\boldsymbol{m} \cdot \boldsymbol{\phi}) \tag{2.5}$$

for all weights $\boldsymbol{m}$ of G and all class parameter vectors $\boldsymbol{\phi}$ of H. This leads, via (2.2) and its analogue for the characters of H, to the recovery of (2.1). The action of P on any particular weight vector $\boldsymbol{m}$ of G is to project it from the weight space of G to that of H. The only difference between P and $\mathscr{P}$ arises from the fact that the definition of P involves, for convenience, the embedding of the weight spaces of G and H of dimension k_g and k_h in Euclidean spaces of dimension d_G and d_H respectively. Thus P is a $d_H \times d_G$ matrix whilst $\mathscr{P}$ is a $k_h \times k_g$ matrix. Since it is the matrix Q which serves to define the relationship between characters of G and H it is this $d_H \times d_G$ matrix which is used in what follows to specify subjoinings.

Following Patera and Sharp (1980) it is convenient to write in place of (2.1) a symbolic form of (2.3), namely

$$\mathrm{G} > \mathrm{H} \qquad \boldsymbol{\lambda}_G \to \sum_{\boldsymbol{\mu}_H} B_{\boldsymbol{\mu}_H}^{\boldsymbol{\lambda}_G} \boldsymbol{\mu}_H. \tag{2.6}$$

Since the definition of subjoining involves sets of weights associated with irreducible representations it follows that for a given subjoining Q is not unique. This is best seen by noting the action of the Weyl symmetry groups which are the symmetry groups of the weight diagrams of irreducible representations. The actions of typical elements S and T of the Weyl symmetry groups $\mathrm{W_G}$ and $\mathrm{W_H}$ of G and H respectively are such that (King and Al-Qubanchi 1980, King 1980)

$$\chi_{S\boldsymbol{\psi}}^{\boldsymbol{\lambda}_G} = \chi_{\boldsymbol{\psi}}^{\boldsymbol{\lambda}_G} \qquad \text{and} \qquad M_{S\boldsymbol{m}}^{\boldsymbol{\lambda}_G} = M_{\boldsymbol{m}}^{\boldsymbol{\lambda}_G} \qquad \text{for } S \in \mathrm{W_G} \tag{2.7}$$

$$\chi_{T\boldsymbol{\phi}}^{\boldsymbol{\mu}_H} = \chi_{\boldsymbol{\phi}}^{\boldsymbol{\mu}_H} \qquad \text{and} \qquad M_{T\boldsymbol{m}}^{\boldsymbol{\mu}_H} = M_{\boldsymbol{m}}^{\boldsymbol{\mu}_H} \qquad \text{for} \qquad T \in \mathrm{W_H}. \tag{2.8}$$

It follows that the operator Q appearing in (2.3) may undergo the transformation

$$Q \to SQT \qquad \text{for} \qquad S \in \mathrm{W_G},\ T \in \mathrm{W_H} \tag{2.9}$$

without affecting (2.3) in any way.

In order to specify subjoinings of H to G which are not equivalent with respect to transformations of the form (2.9) it is not necessary to define the branching multiplicities $B_{\boldsymbol{\mu}_H}^{\boldsymbol{\lambda}_G}$ for all $\boldsymbol{\mu}_H$ in (2.3) and (2.6) for all $\boldsymbol{\lambda}_G$. It is only necessary to give these branching multiplicities for one or, at most, two irreducible representations $\boldsymbol{\lambda}_G$ of G. The criterion which must be fulfilled is that the branching multiplicities should be specified for a sufficiently large, but minimal number of irreducible representations $\boldsymbol{\lambda}_G$

of G for the projection Pm of every possible weight vector of G to be found by making use of the linearity of the action of P. This is entirely analogous to the problem, discussed and solved by Dynkin (1957, page 125), of enumerating embeddings, rather than subjoinings, of H in G defined up to linear equivalence. Thus for both embeddings and subjoinings it is sufficient in almost all cases to determine P, and hence Q, up to equivalence under (2.9) by fixing the branching multiplicities $B^{\omega_G}_{\mu_H}$ for all μ_H, where ω_G is the defining representation of G. In the case of the groups SO($2k$) it is also necessary to specify the branching of one of the spin representations Δ_+.

In the cases for which Q is fixed by the branching of the defining representation ω_G of G into irreducible representations of H it is convenient to write

$$Q = Q_{GH} = Q(q_H) \tag{2.10}$$

where, in the notation of (2.6), the subjoining is defined by

$$G > H \qquad \omega_G \to q_H = \sum_{\mu_H} B^{\omega_G}_{\mu_H} \mu_H. \tag{2.11}$$

As pointed out by Patera *et al* (1980), just as there exist chains of group embeddings so there exist chains of group subjoinings. A subjoining G > H defined by Q_{GH} is said to be maximal if there exist no subjoinings G > K and K > H defined by Q_{GK} and Q_{KH} respectively, such that

$$Q_{GH} = Q_{GK} Q_{KH}, \tag{2.12}$$

other than those for which K is isomorphic to G or H with Q_{GK} or Q_{KH} defining merely an automorphism of G or H respectively.

If the subjoining G > H is not maximal then there does exist a subjoining chain G > K > H such that in the notation of (2.3) and (2.12)

$$\begin{aligned} \chi^{\lambda_G}_{\psi} &= \chi^{\lambda_G}_{Q_{GH}\phi} = \chi^{\lambda_G}_{Q_{GK}Q_{KH}\phi} = \chi^{\lambda_G}_{Q_{GK}\theta} \\ &= \sum_{\nu_K} B^{\lambda_G}_{\nu_K} \chi^{\nu_K}_{\theta} = \sum_{\nu_K} B^{\lambda_G}_{\nu_K} \chi^{\nu_K}_{Q_{KH}\phi} \\ &= \sum_{\nu_K, \mu_H} B^{\lambda_G}_{\nu_K} B^{\nu_K}_{\mu_H} \chi^{\mu_H}_{\phi} \end{aligned} \tag{2.13}$$

so that

$$B^{\lambda_G}_{\mu_H} = \sum_{\nu_K} B^{\lambda_G}_{\nu_K} B^{\nu_K}_{\mu_H}. \tag{2.14}$$

It has been assumed that the class parameters of G, K and H are ψ, θ and ϕ respectively with

$$\psi = Q_{GK}\theta \qquad \text{and} \qquad \theta = Q_{KH}\phi. \tag{2.15}$$

3. Subjoinings specified by S_r

In order to understand the nature of the subjoining relation and to make progress with the problem of enumerating maximal subjoinings it is instructive to consider the unitary groups in some detail.

Each $n\times n$ matrix A which is an element of $\mathrm{U}(n)$ is conjugate to a diagonal matrix with diagonal elements $\exp(\mathrm{i}\phi_1), \exp(\mathrm{i}\phi_2), \ldots, \exp(\mathrm{i}\phi_n)$. Thus the defining irreducible representation of $\mathrm{U}(n)$, denoted by $\{1\}$ has character

$$\chi_{\boldsymbol{\phi}}^{\{1\}}=\operatorname{Tr} A=\exp(\mathrm{i}\phi_1)+\exp(\mathrm{i}\phi_2)+\ldots+\exp(\mathrm{i}\phi_n). \tag{3.1}$$

This is manifestly invariant under permutations of the components of $\boldsymbol{\phi}$ corresponding to the fact that the Weyl symmetry group of $\mathrm{U}(n)$ is the symmetric group S_n.

Each irreducible, covariant tensor representation $\{\boldsymbol{\mu}\}$ of $\mathrm{U}(n)$ is specified by means of a partition $\boldsymbol{\mu}=(\mu_1, \mu_2, \ldots, \mu_n)$ with $\mu_1\geqslant\mu_2\geqslant\ldots\geqslant\mu_n\geqslant 0$. If $\boldsymbol{\mu}$ is a partition of m so that $\mu_1+\mu_2+\ldots+\mu_n=m$ then it is convenient to write $\boldsymbol{\mu}\vdash m$ following the notation of Hall (1959). Littlewood (1950, page 188) showed that

$$\chi_{\boldsymbol{\phi}}^{\{\mu\}}=e_{\boldsymbol{\mu}}(\boldsymbol{x}) \tag{3.2}$$

where

$$x_j=\exp(\mathrm{i}\phi_j) \qquad \text{for} \qquad j=1,2,\ldots,n, \tag{3.3}$$

and the notation is again that of Hall (1959) whereby $e_{\boldsymbol{\mu}}(\boldsymbol{x})$ denotes a Schür function or S function of the indeterminates $x_1, x_2, \ldots, x_n$. This has a combinatorial definition in terms of standard Young tableaux as given by Hall (1959) and Stanley (1971) which may be used (King and Plunkett 1976) to show that $e_{\boldsymbol{\mu}}(\boldsymbol{x})$ is indeed a symmetric function of the components of $\boldsymbol{x}$. If it is to be understood that the indeterminates $\boldsymbol{x}$ are related to class parameters $\boldsymbol{\phi}$ as in (3.3), so that $e_{\boldsymbol{\mu}}(\boldsymbol{x})$ is a unitary group character, it is convenient and conventional to denote this character (3.2) more simply by

$$\{\boldsymbol{\mu}\}=e_{\boldsymbol{\mu}}. \tag{3.4}$$

The simplest symmetric functions are the power-sum symmetric functions

$$\mathrm{S}_r(\boldsymbol{x})=\sum_{j=1}^{n} x_j^r. \tag{3.5}$$

Further symmetric functions are defined multiplicatively by

$$\mathrm{S}_{\boldsymbol{\rho}}(\boldsymbol{x})=\mathrm{S}_{\rho_1}(\boldsymbol{x})\mathrm{S}_{\rho_2}(\boldsymbol{x})\ldots \tag{3.6}$$

where $\boldsymbol{\rho}=(\rho_1, \rho_2, \ldots)$ is a partition. Such a function serves as a generating function for characters $\chi_{\boldsymbol{\rho}}^{\boldsymbol{\mu}}$ of the symmetric group S_r if $\boldsymbol{\rho}$ is a partition of r, through the relation due to Frobenius (1900):

$$\mathrm{S}_{\boldsymbol{\rho}}(\boldsymbol{x})=\sum_{\boldsymbol{\mu}\vdash r}\chi_{\boldsymbol{\rho}}^{\boldsymbol{\mu}}\, e_{\boldsymbol{\mu}}(\boldsymbol{x}). \tag{3.7}$$

A particularly important special case of this relation, applicable to (3.3), follows from a result of Murnaghan (1938, page 134) and takes the form

$$\mathrm{S}_r(\boldsymbol{x})=\sum_{b=0}^{r-1}(-1)^b\, e_{r-b,1^b}(\boldsymbol{x}). \tag{3.8}$$

With this preamble it is easy to interpret a whole class of subjoinings to the unitary group analogous to those discussed by Patera *et al* (1980) for which $\mathscr{P}$ is a multiple of the unit matrix. To be specific the subjoining $\mathrm{U}(n)>\mathrm{U}(n)$ corresponding to

$$Q=P=rI \tag{3.9}$$

is defined by

$$\chi_{\psi}^{\{1\}} = \chi_{r\phi}^{\{1\}}, \tag{3.10}$$

so that

$$\begin{aligned}\chi_{\psi}^{\{1\}} &= \exp(i\psi_1)+\exp(i\psi_2)+\ldots+\exp(i\psi_n)\\ &= \exp(ir\phi_1)+\exp(ir\phi_2)+\ldots+\exp(ir\phi_n)\\ &= S_r(\boldsymbol{x}) = \sum_{b=0}^{\min(r-1,n-1)} (-1)^b \chi_{\boldsymbol{\phi}}^{\{r-b,1^b\}},\end{aligned} \tag{3.11}$$

where use has been made successively of (3.1), (3.10), (3.5), (3.8) and (3.2).

This result (3.11) specifies the subjoining of U(n) to U(n) defined by

$$Q = Q(S_r) = rI \tag{3.12}$$

and may conveniently be written in the symbolic form

$$\begin{aligned}U(n)>U(n) \qquad & \{1\}\to S_r = \exp(ir\phi_1)+\exp(ir\phi_2)+\ldots+\exp(ir\phi_n)\\ &= \{r\}-\{r-1,1\}+\{r-2,1^2\}-\ldots.\end{aligned} \tag{3.13}$$

The corresponding branching rule (2.6) is given by

$$U(n)>U(n) \qquad \{\boldsymbol{\lambda}\}\to S_r\otimes\{\boldsymbol{\lambda}\} \tag{3.14}$$

where $\otimes$ signifies the operation of plethysms introduced by Littlewood (1950, page 206) as a new multiplication of S functions. This is essentially a type of substitutional operation which applies to any symmetric functions, so that in particular

$$S_r\otimes\{\boldsymbol{\lambda}\} = S_r\otimes e_{\boldsymbol{\lambda}} = e_{\boldsymbol{\lambda}}(x_1^r, x_2^r, \ldots) = e_{\boldsymbol{\lambda}}\otimes S_r = \{\boldsymbol{\lambda}\}\otimes S_r. \tag{3.15}$$

This identity allows the evaluation of the branching rule (3.14) to be made via the combinatorial methods of Littlewood (1951), Foulkes (1951) or Plunkett (1972). Alternatively, the use of (3.15) and (3.8) in (3.14) gives

$$\begin{aligned}U(n)>U(n) \qquad & \{\boldsymbol{\lambda}\}\to(\{r\}-\{r-1,1\}+\{r-2,1^2\}-\ldots)\otimes\{\boldsymbol{\lambda}\}\\ &= \{\boldsymbol{\lambda}\}\otimes(\{r\}-\{r-1,1\}+\{r-2,1^2\}-\ldots)\\ &= \sum_{\boldsymbol{\mu}} B_{\{\boldsymbol{\mu}\}}^{\{\boldsymbol{\lambda}\}}\{\boldsymbol{\mu}\}\end{aligned} \tag{3.16}$$

where the branching multiplicities $B_{\{\boldsymbol{\mu}\}}^{\{\boldsymbol{\lambda}\}}$ may now be evaluated using the algebra of plethysms developed by Littlewood (1950, page 290).

An illustrative example is provided by the case $r=3$, giving in place of (3.13)

$$U(n)>U(n) \qquad \{1\}\to S_3 = \{3\}-\{21\}+\{1^3\}. \tag{3.17}$$

It follows that

$$\begin{aligned}U(n)>U(n) \qquad & \{2\}\to S_3\otimes\{2\} = \{2\}\otimes S_3\\ &= \{2\}\otimes(\{3\}-\{21\}+\{1^3\})\\ &= \{2\}\otimes\{3\}-\{2\}\otimes\{21\}+\{2\}\otimes\{1^3\}\\ &= \{6\}+\{42\}+\{2^3\}-\{51\}-\{42\}-\{321\}+\{3^2\}+\{41^2\}\\ &= \{6\}-\{51\}+\{41^2\}+\{3^2\}-\{321\}+\{2^3\}\end{aligned} \tag{3.18}$$

and

$$\{1^2\} \to S_3 \otimes \{1^2\} = \{1^2\} \otimes S_3 = \{3^2\} - \{321\} + \{31^3\} + \{2^3\} - \{21^4\} + \{1^6\}, \tag{3.19}$$

where use has been made of the known plethysms of the form $\{\nu\} \otimes \{\boldsymbol{\lambda}\}$ as tabulated, for example, by Butler and Wybourne (1971).

In the case of the subjoining defined by (3.9), or equivalently (3.13), the integer r need not necessarily be prime. However, as pointed out by Patera *et al* (1980), such a subjoining is only maximal if this is the case. Indeed if $P = Q = rI = pqI$ it is clear that the subjoining corresponding to $\boldsymbol{\psi} = pq\boldsymbol{\phi}$, defined by

$$U(n) > U(n) \qquad \{1\} \to S_r = S_{pq} = \exp(ipq\phi_1) + \exp(ipq\phi_2) + \ldots + \exp(ipq\phi_n) \tag{3.20}$$

may be defined by the successive transformations $\boldsymbol{\theta} = q\boldsymbol{\phi}$ and $\boldsymbol{\psi} = p\boldsymbol{\theta}$ corresponding to the existence of the subjoining chain $U(n) > U(n) > U(n)$ with the first stage defined by $Q = Q(S_p) = pI$ and the second by $Q = Q(S_q) = qI$ so that in accordance with (2.12), (3.20) is defined by

$$Q = Q(S_p)Q(S_q) = pIqI = pqI = Q(S_{pq}). \tag{3.21}$$

It follows that the complete chain is specified by

$$U(n) > U(n) > U(n) \qquad \{1\} \to S_p \to S_{pq}$$

and the corresponding branching rule is then

$$\begin{aligned} U(n) > U(n) > U(n) \qquad & \{\boldsymbol{\lambda}\} \to S_p \otimes \{\boldsymbol{\lambda}\} \to S_q \otimes (S_p \otimes \{\boldsymbol{\lambda}\}) \\ & = (S_q \otimes S_p) \otimes \{\boldsymbol{\lambda}\} \\ & = S_{pq} \otimes \{\boldsymbol{\lambda}\} \end{aligned}$$

where use has been made of the associativity of the operation $\otimes$ to demonstrate that the result is in accordance with that which may be obtained directly from (3.20).

4. Subjoinings specified by S_ρ

Further subjoinings may now be generated by making use of (3.6) to define the subjoining of $U(n)$ to $U(n^p)$ specified by

$$U(n^p) > U(n) \qquad \{1\} \to S_{\boldsymbol{\rho}} = \sum_{\boldsymbol{\mu} \vdash r} \chi^{\mu}_{\boldsymbol{\rho}} \{\boldsymbol{\mu}\} \tag{4.1}$$

where $\boldsymbol{\rho}$ is a partition of r into p non-vanishing parts and use has been made of (3.7) and (3.4). The corresponding branching rule takes the form

$$U(n^p) > U(n) \qquad \{\boldsymbol{\lambda}\} \to S_{\boldsymbol{\rho}} \otimes \{\boldsymbol{\lambda}\} = \Big(\sum_{\boldsymbol{\mu} \vdash r} \chi^{\mu}_{\boldsymbol{\rho}} \{\boldsymbol{\mu}\} \Big) \otimes \{\boldsymbol{\lambda}\}. \tag{4.2}$$

As an example, the case $\boldsymbol{\rho} = (2, 1)$ defines the subjoining

$$U(n^2) > U(n) \qquad \{1\} \to S_{2,1} = \sum_{\boldsymbol{\mu} \vdash 3} \chi^{\mu}_{21} \{\boldsymbol{\mu}\} = \{3\} - \{1^3\} \tag{4.3}$$

corresponding, in more detail, to

$$\{1\} = \exp(i\psi_1) + \exp(i\psi_2) + \ldots + \exp(i\psi_{n^2})$$
$$\to S_{2,1}(\boldsymbol{x}) = (x_1^2 + x_2^2 + \ldots + x_n^2)(x_1 + x_2 + \ldots + x_n)$$
$$= (\exp(i2\phi_1) + \exp(i2\phi_2) + \ldots + \exp(i2\phi_n))(\exp(i\phi_1) + \exp(i\phi_2) + \ldots + \exp(i\phi_n))$$
$$= \exp(i3\phi_1) + \exp[i(2\phi_1 + \phi_2)] + \exp[i(2\phi_1 + \phi_3)] + \ldots . \tag{4.4}$$

Thus with a suitable ordering of these terms the $n^2 \times n$ matrix Q is given by

$$Q = Q(S_{2,1}) = \begin{bmatrix} 3 & 0 & 0 & . & . & . & 0 \\ 2 & 1 & 0 & . & . & . & 0 \\ 2 & 0 & 1 & . & . & . & 0 \\ \vdots & & & & & & \\ 2 & 0 & 0 & . & . & . & 1 \\ 1 & 2 & 0 & . & . & . & 0 \\ 0 & 3 & 0 & . & . & . & 0 \\ 0 & 2 & 1 & . & . & . & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & . & . & . & 3 \end{bmatrix} . \tag{4.5}$$

A typical branching then takes the form

$$U(n^2) > U(n) \qquad \{2\} \to S_{2,1} \otimes \{2\} = (\{3\} - \{1^3\}) \otimes \{2\}$$
$$= \{3\} \otimes \{2\} - \{3\} . \{1^3\} + \{1^3\} \otimes \{1^2\}$$
$$= \{6\} + \{42\} - \{41^2\} - \{31^3\} + \{2^2 1^2\} + \{1^6\}. \tag{4.6}$$

This subjoining is clearly not maximal since it may be defined by the chain:

$$U(n^2) \supset U(n) \times U(n) > U(n) \times U(n) \supset U(n)$$
$$\{1\} \to \{1\} \times \{1\} \to S_2 \times S_1 \to S_2 S_1 = S_{2,1} \tag{4.7}$$

where

$$S_2 = \{2\} - \{1^2\} \qquad \text{and} \qquad S_1 = \{1\} \tag{4.8}$$

so that for $U(n^2) \supset U(n) \times U(n) > U(n) \times U(n) \supset U(n)$

$$\{1\} \to \{1\} \times \{1\} \to (\{2\} - \{1^2\}) \times \{1\} \to (\{2\} - \{1^2\}) . \{1\}$$
$$= \{3\} + \{21\} - \{21\} - \{1^3\} = \{3\} - \{1^3\}, \tag{4.9}$$

in agreement with (4.3). The use of the chain (4.7) together with (4.8) then gives the branching

$$\{2\} \to \{2\} \times \{2\} + \{1^2\} \times \{1^2\} \to [(\{2\} - \{1^2\}) \otimes \{2\}] \times [\{1\} \otimes \{2\}]$$
$$+ [(\{2\} - \{1^2\}) \otimes \{1^2\}] \times [\{1\} \otimes \{1^2\}]$$
$$= (\{4\} - \{31\} + \{2^2\}) \times \{2\} + (\{2^2\} - \{21^2\} + \{1^4\}) \times \{1^2\}$$
$$\to \{6\} + \{51\} + \{42\} - \{51\} - \{42\} - \{41^2\} - \{3^2\} + \{42\} + \{321\} + \{2^3\}$$
$$+ \{3^2\} + \{321\} + \{2^2 1^2\} - \{321\} - \{31^3\} - \{2^3\} - \{2^2 1^2\}$$

$$-\{21^4\}+\{2^21^2\}+\{21^4\}+\{1^6\}$$
$$=\{6\}+\{42\}-\{41^2\}-\{31^3\}+\{2^21^2\}+\{1^6\}, \tag{4.10}$$

in agreement with (4.6).

The non-maximal nature of the subjoining (4.3), as shown by the existence of the chain (4.7), is also illustrated by the factorisation of $Q(\mathrm{S}_{2,1})$, given in (4.5), in the form

$$Q(\mathrm{S}_{2,1})=$$
$$\begin{bmatrix} 1 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots \\ 1 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \\ 0 & 1 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots \\ 0 & 1 & \dots & 0 & 0 & 0 & \dots & 1 \\ \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 2 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots \\ 0 & 0 & \dots & 2 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots \\ 0 & 0 & \dots & 1 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$
$$=Q(\{1\}\times\{1\})Q(\mathrm{S}_2\otimes\{0\}+\{0\}\times\mathrm{S}_1)Q(\{1\}+\{1\}) \tag{4.11}$$

where the three factors correspond to the subjoinings

$$\mathrm{U}(n^2)\supset\mathrm{U}(n)\times\mathrm{U}(n) \qquad \{1\}\to\{1\}\times\{1\} \tag{4.12}$$
$$\mathrm{U}(n)\times\mathrm{U}(n)>\mathrm{U}(n)\times\mathrm{U}(n) \qquad \{1\}\times\{0\}+\{0\}\times\{1\}\to\mathrm{S}_2\times\{0\}+\{0\}\times\mathrm{S}_1 \tag{4.13}$$
$$\mathrm{U}(n)\times\mathrm{U}(n)\supset\mathrm{U}(n) \qquad \{1\}\times\{0\}+\{0\}\times\{1\}\to\{1\}+\{1\}. \tag{4.14}$$

These in turn yield the branching rules, used already in evaluating (4.10),

$$\mathrm{U}(n^2)\supset\mathrm{U}(n)\supset\mathrm{U}(n) \qquad \{\boldsymbol{\lambda}\}\to\sum_{\boldsymbol{\mu},\boldsymbol{\nu}} k^{\lambda}_{\mu\nu}\{\boldsymbol{\mu}\}\times\{\boldsymbol{\nu}\} \tag{4.15}$$
$$\mathrm{U}(n)\times\mathrm{U}(n)>\mathrm{U}(n)\times\mathrm{U}(n) \qquad \{\boldsymbol{\mu}\}\times\{\boldsymbol{\nu}\}\to(\mathrm{S}_2\otimes\{\boldsymbol{\mu}\})(\mathrm{S}_1\otimes\{\boldsymbol{\nu}\})$$
$$=(\{\boldsymbol{\mu}\}\otimes\mathrm{S}_2)\times\{\boldsymbol{\nu}\} \tag{4.16}$$
$$\mathrm{U}(n)\times\mathrm{U}(n)\supset\mathrm{U}(n) \qquad \{\boldsymbol{\sigma}\}\times\{\boldsymbol{\tau}\}\to\sum_{f} m^{\rho}_{\sigma\tau}\{\boldsymbol{\rho}\}. \tag{4.17}$$

The coefficients $k^{\lambda}_{\mu\nu}$ are the Kronecker-product multiplicities of S_m defined by

$$\chi^{\mu}_{\rho}\chi^{\nu}_{\rho}=\sum_{\lambda} k^{\lambda}_{\mu\nu}\chi^{\lambda}_{\rho} \tag{4.18}$$

where $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, $\boldsymbol{\nu}$ and $\boldsymbol{\rho}$ are all partitions of m. Similarly, the coefficients $m^{\rho}_{\sigma\tau}$ are the Kronecker-product multiplicities of $\mathrm{U}(n)$ defined by the S-function product

$$\{\boldsymbol{\sigma}\}\{\boldsymbol{\tau}\}=\sum_{\mu} m^{\rho}_{\sigma\tau}\{\boldsymbol{\rho}\} \tag{4.19}$$

where $\boldsymbol{\rho}$ is a partition of $r=s+t$ if $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ are partitions of s and t respectively.

Just as (4.3) is not maximal, so for $p>1$ the subjoining (4.2) is not maximal as indicated by the existence of the chain

$$\begin{gathered}U(n^p)\supset U(n)\times U(n)\times\ldots\times U(n)>U(n)\times\ldots\times U(n)\supset U(n)\\ \{1\}\to\{1\}\times\{1\}\times\ldots\times\{1\}\to S_{\rho_1}\times S_{\rho_2}\times\ldots\times S_{\rho_p}\to S_{\rho_1}S_{\rho_2}\ldots S_{\rho_p}=S_{\boldsymbol{\rho}}.\end{gathered}\tag{4.20}$$

Clearly the non-maximal nature of the subjoining is a consequence of the multiplicative nature of the symmetric function $S_{\boldsymbol{\rho}}(\boldsymbol{x})$.

5. Subjoinings specified by k_ρ

To determine inequivalent maximal subjoinings other than those of § 3 it is important to realise the nature of the transformations (2.9) with respect to which subjoinings are equivalent. These transformations on Q imply, through (2.4), that for the subjoining $U(N)>U(n)$ the N weights of the defining representation $\{1\}$ of $U(N)$, which are generated from the highest weight $(1, 0, 0, \ldots, 0)$ by the elements S of $W_{U(N)}=S_N$, are mapped by P onto a set of weights forming the basis of a permutation representation of $W_{U(n)}=S_n$, in the sense that the action of each element T of S_n on a weight in this set yields, in general, another weight in the set. If the subjoining is to be maximal then the highest weight of this set must be unique and the whole set must be obtainable from this highest weight by the action of elements of S_n. Such a weight is said to be $U(n)$-dominant (King and Al-Qubanchi 1980) and takes the form $\boldsymbol{\rho}=(\rho_1, \rho_2, \ldots, \rho_n)$ where $\boldsymbol{\rho}$ is a partition. The complete set of weights said to be $U(n)$-equivalent to $\boldsymbol{\rho}$ is defined by

$$C_{\boldsymbol{\rho}}=\{\boldsymbol{m}:\boldsymbol{m}=T\boldsymbol{\rho},\ T\in W_{U(n)}=S_n\}.\tag{5.1}$$

In the case $\boldsymbol{\rho}=(\rho_1, \rho_2, \ldots, \rho_n)=(\ldots 2^{\alpha_2}1^{\alpha_1}0^{\alpha_0})$ with $n=\alpha_0+\alpha_1+\alpha_2+\ldots$, the number of distinct elements in $C_{\boldsymbol{\rho}}$ is given by

$$c_\rho=n!/\alpha_0!\alpha_1!\alpha_2!\ldots.\tag{5.2}$$

It follows that maximal subjoinings are specified by

$$U(c_{\boldsymbol{\rho}})>U(n)\qquad \{1\}\to\sum_{\boldsymbol{m}\in C_{\boldsymbol{\rho}}}\exp(\mathrm{i}\boldsymbol{m}\cdot\boldsymbol{\phi}).\tag{5.3}$$

In the special case where $\boldsymbol{\rho}=(r, 0, 0, \ldots, 0)$ and, correspondingly, $c_{\boldsymbol{\rho}}=n$ it is clear that (5.3) reduces to (3.13).

More generally, adopting the notation of (3.3) it follows that the right-hand side of (5.3) is nothing other than the monomial symmetric function

$$k_{\boldsymbol{\rho}}(\boldsymbol{x})=\sum_{\pi}x_1^{\rho_{\pi_1}}x_2^{\rho_{\pi_2}}\ldots x_n^{\rho_{\pi_n}}\tag{5.4}$$

where, as in the definitions of Littlewood (1950, page 63), Hall (1959) and Stanley (1971), the summation is over those permutations $\pi\in S_n$ leading to distinct monomials in the components of $\boldsymbol{x}$. The analogue of (3.7) is the expansion

$$k_{\boldsymbol{\rho}}(\boldsymbol{x})=\sum_{\mu\vdash r}L^{\mu}_{\boldsymbol{\rho}}e_{\mu}(\boldsymbol{x})\tag{5.5}$$

which may be effected by the methods of Murnaghan (1938, page 163) or Littlewood (1950, page 90).

Thus (5.3) may be written in the form

$$\mathrm{U}(c_{\rho})>\mathrm{U}(n) \qquad \{1\}\to k_{\rho}=\sum_{\mu\vdash r} L^{\mu}_{\rho}\{\mu\} \tag{5.6}$$

where ρ is a partition of r. The corresponding branching rule takes the form

$$\mathrm{U}(c_{\rho})>\mathrm{U}(n) \qquad \{\lambda\}\to k_{\rho}\otimes\{\lambda\}=\Big(\sum_{\mu\vdash r} L^{\mu}_{\rho}\{\mu\}\Big)\otimes\{\lambda\}, \tag{5.7}$$

which yields the branching multiplicity coefficients through the algebra of plethysms.

An example of a maximal subjoining of this type is provided by

$$\begin{aligned}\mathrm{U}(n(n-1))>\mathrm{U}(n) \qquad \{1\}&\to k_{21}(\boldsymbol{x})\\ &=x_1^2x_2+x_1^2x_3+\ldots+x_1^2x_n+x_2^2x_1+x_2^2x_3+\ldots\\ &=e_{21}(\boldsymbol{x})-2e_{1^3}(\boldsymbol{x})=\{21\}-2\{1^3\}.\end{aligned} \tag{5.8}$$

This corresponds to

$$\begin{aligned}\chi^{\{1\}}_{\psi}&=\exp(\mathrm{i}\psi_1)+\exp(\mathrm{i}\psi_2)+\ldots+\exp(\mathrm{i}\psi_{n(n-1)})\\ &=\exp[\mathrm{i}(2\phi_1+\phi_2)]+\exp[\mathrm{i}(2\phi_1+\phi_3)]+\ldots+\exp[\mathrm{i}(2\phi_1+\phi_n)]\\ &\quad+\exp[\mathrm{i}(\phi_1+2\phi_2)]+\exp[\mathrm{i}(2\phi_2+\phi_3)]+\ldots\\ &=\chi^{\{21\}}_{\phi}-2\chi^{\{1^3\}}_{\phi},\end{aligned} \tag{5.9}$$

so that

$$Q=Q(k_{21})=\begin{bmatrix}2&1&0&\ldots&0&0\\ 2&0&1&\ldots&0&0\\ \vdots&&&&&\vdots\\ 2&0&0&\ldots&0&1\\ 1&2&0&\ldots&0&0\\ 0&2&1&\ldots&0&0\\ \vdots&&&&&\vdots\\ 0&0&0&\ldots&1&2\end{bmatrix}. \tag{5.10}$$

A typical branching is given by

$$\mathrm{U}(n(n-1))>\mathrm{U}(n) \qquad \{2\}\to k_{21}\otimes\{2\}=(\{21\}-2\{1^3\})\otimes\{2\} \tag{5.11}$$

$$\begin{aligned}&=\{21\}\otimes\{2\}+2(\{1^3\}\otimes\{1^2\})-2(\{21\}\,.\,\{1^3\})+\{1^3\}\,.\,\{1^3\}\\ &=\{42\}-\{321\}-\{31^3\}+2\{2^3\}+\{2^21^2\}-\{21^4\}+3\{1^6\}.\end{aligned} \tag{5.12}$$

Two further examples of maximal subjoinings of the form (5.6) are provided by

$$\mathrm{U}(n)>\mathrm{U}(n) \qquad \{1\}\to k_3=S_3=\{3\}-\{21\}+\{1^3\} \tag{5.13}$$

and

$$\mathrm{U}(\tfrac{1}{6}n(n-1)(n-2))\supset\mathrm{U}(n) \qquad \{1\}\to k_{1^3}=e_{1^3}=\{1^3\}. \tag{5.14}$$

The former coincides with (3.17) and yields the branching (3.18), whilst the latter gives

$$\mathrm{U}(\tfrac{1}{6}n(n-1)(n-2)) \supset \mathrm{U}(n) \qquad \{2\} \to k_{1^3} \otimes \{2\} = \{1^3\} \otimes \{2\}$$
$$= \{2^3\} + \{21^4\}. \tag{5.15}$$

Furthermore, it is easy to see that (4.3) may be recovered by combining (5.12) and (5.8) to give the subjoining chain

$$\mathrm{U}(n^2) \supset \mathrm{U}(n) \times \mathrm{U}(n(n-1)) > \mathrm{U}(n) \times \mathrm{U}(n) \supset \mathrm{U}(n)$$
$$\{1\} \to \{1\} \times \{0\} + \{0\} \times \{1\} \to k_3 \times \{0\} + \{0\} \times k_{21} \to k_3 + k_{21}$$
$$= \{3\} - \{21\} + \{1^3\} + \{21\} - 2\{1^3\} = \{3\} - \{1^3\} = \mathrm{S}_{21}. \tag{5.16}$$

The corresponding branching rules yield

$$\begin{aligned}\{2\} \to \{2\} \times \{0\} + \{1\} \times \{1\} + \{0\} \times \{2\} &\to (k_3 \otimes \{2\}) \times \{0\} + k_3 \times k_{21} + \{0\} \times (k_{21} \otimes \{2\}) \\ &= (\{6\} - \{51\} + \{41^2\} + \{3^2\} - \{321\} + \{2^3\}) \times \{0\} \\ &\quad + (\{3\} - \{21\} + \{1^3\}) \times (\{21\} - 2\{1^3\}) + \{0\} \times (\{42\} - \{321\} - \{31^3\} + 2\{2^3\} \\ &\quad + \{2^2 1^2\} - \{21^4\} + 3\{1^6\}) \\ &\to \{6\} + \{42\} - \{41^2\} - \{31^3\} + \{2^2 1^2\} + \{1^6\}\end{aligned} \tag{5.17}$$

where use has been made of (3.18) and (5.12) to obtain, once more, agreement with (4.6). In addition use has been made of the fact that the first stage is defined by the subjoining

$$\mathrm{U}(n^2) \supset \mathrm{U}(n) \times \mathrm{U}(n(n-1)) \qquad \{1\} \to \{1\} \times \{0\} + \{0\} \times \{1\} \tag{5.18}$$

with branching rule

$$\mathrm{U}(n^2) \supset \mathrm{U}(n) \times \mathrm{U}(n(n-1)) \qquad \{\lambda\} \to \sum_{\lambda} m^{\lambda}_{\sigma\tau} \{\sigma\} \times \{\tau\} \tag{5.19}$$

and the last stage is defined by (4.14) with branching rule (4.17), so that the relevant coefficients are given in both cases by (4.19).

The non-maximal nature of (4.3) is thus illustrated not just by the existence of the subjoining chain (4.7) but also by the existence of the subjoining chain (5.16). In the latter case the corresponding factorisation of $Q(\mathrm{S}_{21})$ is given by

$$Q(\mathrm{S}_{21}) =$$

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & & & & \vdots \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 3 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & & & & & \\ 0 & 0 & \cdots & 3 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 2 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 2 & 2 & 0 & 1 & \cdots & 0 \\ \vdots & & & & & & & & \vdots \\ 0 & 0 & \cdots & 0 & 2 & 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 1 & 2 & 0 & \cdots & 0 \\ \vdots & & & & & & & & \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$= Q(\{1\} \times \{0\} + \{0\} \times \{1\}) Q(k_3 \times \{0\} + \{0\} \times k_{21}) Q(\{1\} + \{1\}). \tag{5.20}$$

The matrix $Q(S_{21})$ formed from this product is equivalent to the matrix formed from the product (4.11) under the transformation of the form (2.9) in which S permutes the rows appropriately and T is the identity.

More generally, the non-maximal nature of (4.2) is illustrated not just by the existence of the chain (4.20) but also by the existence of the chain

$$\begin{aligned}
&\mathrm{U}(n^p) \supset \mathrm{U}(c_{\nu(1)}) \times \mathrm{U}(c_{\nu(2)}) \times \mathrm{U}(c_{\nu(3)}) \times \ldots > \mathrm{U}(n) \times \mathrm{U}(n) \times \mathrm{U}(n) \times \ldots \supset \mathrm{U}(n) \\
&\quad \{1\} \to \{1\} \times \{0\} \times \{0\} \times \ldots + \{0\} \times \{1\} \times \{0\} \times \ldots + \{0\} \times \{0\} \times \{1\} \times \ldots + \ldots \\
&\quad \to k_{\nu(1)} \times \{0\} \times \{0\} \times \ldots + \{0\} \times k_{\nu(2)} \times \{0\} \times \ldots + \{0\} \times \{0\} \times k_{\nu(3)} \times \ldots + \ldots \\
&\quad \to k_{\nu(1)} + k_{\nu(2)} + k_{\nu(3)} + \ldots = \mathrm{S}_\rho
\end{aligned} \tag{5.21}$$

where the partitions $\boldsymbol{\nu}(1)$, $\boldsymbol{\nu}(2)$, $\boldsymbol{\nu}(3), \ldots$ may be found by making use of the expansion given by Littlewood (1950, page 63):

$$\begin{aligned}
\mathrm{S}_\rho(\boldsymbol{x}) &= \sum_{\nu \vdash r} \phi^\nu_\rho k_\nu(\boldsymbol{x}) \\
&= k_{\nu(1)}(\boldsymbol{x}) + k_{\nu(2)}(\boldsymbol{x}) + k_{\nu(3)}(\boldsymbol{x}) + \ldots
\end{aligned} \tag{5.22}$$

in which the coefficients ϕ^ν_ρ are known compound characters of S_r. The corresponding branching rule takes the form

$$\begin{aligned}
\{\boldsymbol{\lambda}\} &\to \sum_{\sigma(1),\sigma(2),\sigma(3)\ldots} m^{\lambda}_{\sigma(1)\sigma(2)\sigma(3)\ldots} \{\boldsymbol{\sigma}(1)\} \times \{\boldsymbol{\sigma}(2)\} \times \{\boldsymbol{\sigma}(3)\} \times \ldots \\
&\to \sum_{\sigma(1),\sigma(2),\sigma(3)\ldots} m^{\lambda}_{\sigma(1)\sigma(2)\sigma(3)\ldots} (k_{\nu(1)} \otimes \{\boldsymbol{\sigma}(1)\}) \times (k_{\nu(2)} \otimes \{\boldsymbol{\sigma}(2)\}) \times \ldots \\
&\to \sum_{\sigma(1),\sigma(2),\sigma(3)\ldots} m^{\lambda}_{\sigma(1)\sigma(2)\sigma(3)\ldots} (k_{\nu(1)} \otimes \{\boldsymbol{\sigma}(1)\}) \cdot (k_{\nu(2)} \otimes \{\boldsymbol{\sigma}(2)\}) \ldots
\end{aligned} \tag{5.23}$$

where

$$\{\boldsymbol{\sigma}(1)\} \cdot \{\boldsymbol{\sigma}(2)\} \cdot \{\boldsymbol{\sigma}(3)\} \ldots = \sum_{\lambda} m^{\lambda}_{\sigma(1)\sigma(2)\sigma(3)\ldots} \{\boldsymbol{\lambda}\}. \tag{5.24}$$

This illustrates quite clearly the fundamental nature of the subjoinings defined by (5.6) which include all possible maximal subjoinings of $\mathrm{U}(n)$ to $\mathrm{U}(N)$.

However not all such subjoinings are maximal, as can be seen by considering the example

$$\mathrm{U}(n(n-1)) > \mathrm{U}(n) \qquad \{1\} \to k_{42}(\boldsymbol{x}) \tag{5.25}$$

which may be defined by the chain

$$\mathrm{U}(n(n-1)) > \mathrm{U}(n(n-1)) > \mathrm{U}(n) \qquad \{1\} \to k_2 \to k_{42} \tag{5.26}$$

where the first stage is just a particular case of the subjoining (3.13) since $k_2 = \mathrm{S}_2$ and the second stage is defined by (5.8) since

$$k_{21} \otimes k_2 = k_{42}. \tag{5.27}$$

The corresponding factorisation of Q is given by

$$Q(k_{42}) = \begin{bmatrix} 4 & 2 & 0 & \dots & 0 & 0 \\ 4 & 0 & 2 & \dots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \dots & 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 & \dots & 0 \\ 0 & 2 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & \dots & 0 & 0 \\ 2 & 0 & 1 & \dots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \dots & 1 & 2 \end{bmatrix}$$
$$= Q(k_2)Q(k_{21}), \tag{5.28}$$

whilst the branching rule takes the form

$$\mathrm{U}(n(n-1)) > \mathrm{U}(n(n-1)) > \mathrm{U}(n) \qquad \{\boldsymbol{\lambda}\} \to k_2 \otimes \{\boldsymbol{\lambda}\} \to k_{21} \otimes (k_2 \otimes \{\boldsymbol{\lambda}\})$$
$$= (k_{21} \otimes k_2) \otimes \{\boldsymbol{\lambda}\} = k_{42} \otimes \{\boldsymbol{\lambda}\}. \tag{5.29}$$

In general, the subjoining (5.6)

$$\mathrm{U}(c_{\boldsymbol{\rho}}) > \mathrm{U}(n) \qquad \{1\} \to k_{\boldsymbol{\rho}}$$

with $\boldsymbol{\rho} = (\rho_1, \rho_2, \dots, \rho_p)$, is maximal if and only if $\rho_1, \rho_2, \dots$ and ρ_p have no common factor. If, on the other hand, q is the highest common factor so that $\boldsymbol{\rho} = q(\sigma_1, \sigma_2, \dots, \sigma_p)$ then there exists the chain

$$\mathrm{U}(c_{\boldsymbol{\rho}}) = \mathrm{U}(c_{\boldsymbol{\sigma}}) > \mathrm{U}(c_{\boldsymbol{\sigma}}) > \mathrm{U}(n) \qquad \{1\} \to k_q \to k_{\boldsymbol{\rho}} \tag{5.30}$$

with branching rule

$$\{\boldsymbol{\lambda}\} \to k_q \otimes \{\boldsymbol{\lambda}\} \to k_{\boldsymbol{\sigma}} \otimes (k_q \otimes \{\boldsymbol{\lambda}\}) = (k_{\boldsymbol{\sigma}} \otimes k_q) \otimes \{\boldsymbol{\lambda}\}$$
$$= k_{q\boldsymbol{\sigma}} \otimes \{\boldsymbol{\lambda}\} = k_{\boldsymbol{\rho}} \otimes \{\boldsymbol{\lambda}\}. \tag{5.13}$$

6. Subjoinings specified by e_μ

Subjoinings include embeddings as a special case. These are those subjoinings of the type (2.3) in which the branching multiplicities $B^{\boldsymbol{\lambda}_{\mathrm{G}}}_{\boldsymbol{\mu}_{\mathrm{H}}}$ are restricted to be non-negative integers for all irreducible representations $\boldsymbol{\lambda}_{\mathrm{G}}$ and $\boldsymbol{\mu}_{\mathrm{H}}$ of G and H respectively. In such a case it is conventional to write $\mathrm{G} \supset \mathrm{H}$ rather than $\mathrm{G} > \mathrm{H}$.

In the case of the group $\mathrm{G} = \mathrm{U(N)}$ embeddings are defined as in (2.11) by

$$\mathrm{U}(N) \supset \mathrm{H} \qquad \{1\} \to \boldsymbol{q}_{\mathrm{H}} = \sum_{\boldsymbol{\mu}_{\mathrm{H}}} B^{\{1\}}_{\boldsymbol{\mu}_{\mathrm{H}}} \boldsymbol{\mu}_{\mathrm{H}} = \boldsymbol{\mu}_{\mathrm{H}}(1) + \boldsymbol{\mu}_{\mathrm{H}}(2) + \dots, \tag{6.1}$$

where necessarily

$$N = N_1 + N_2 + \dots \tag{6.2}$$

and

$$N_j = d_{\boldsymbol{\mu}_{\mathrm{H}}(j)} \tag{6.3}$$

is the dimension of the irreducible representation $\boldsymbol{\mu}_{\mathrm{H}}(j)$. This embedding is clearly not maximal by virtue of the existence of the chain

$$\begin{aligned} \mathrm{U}(N) \supset \mathrm{U}(N_1) \times \mathrm{U}(N_2) \times \ldots \supset \mathrm{H} \times \mathrm{H} \times \ldots \supset \mathrm{H} \\ \{1\} \to \{1\} \times \{0\} \times \ldots + \{0\} \times \{1\} \times \ldots + \ldots \\ \to \boldsymbol{\mu}_{\mathrm{H}}(1) \times \boldsymbol{\eta}_{\mathrm{H}} \times \ldots + \boldsymbol{\eta}_{\mathrm{H}} \times \boldsymbol{\mu}_{\mathrm{H}}(2) \times \ldots + \ldots \\ \to \boldsymbol{\mu}_{\mathrm{H}}(1) + \boldsymbol{\mu}_{\mathrm{H}}(2) + \ldots \end{aligned} \tag{6.4}$$

where $\boldsymbol{\eta}_{\mathrm{H}}$ denotes the identity representation of H. The corresponding branching rule is:

$$\begin{aligned} \{1\} &\to \sum_{\boldsymbol{\sigma}(1),\boldsymbol{\sigma}(2),\ldots} m^{\boldsymbol{\lambda}}_{\boldsymbol{\sigma}(1)\boldsymbol{\sigma}(2)\ldots} \{\boldsymbol{\sigma}(1)\} \times \{\boldsymbol{\sigma}(2)\} \times \ldots \\ &\to \sum_{\boldsymbol{\sigma}(1),\boldsymbol{\sigma}(2),\ldots} m^{\boldsymbol{\lambda}}_{\boldsymbol{\sigma}(1)\boldsymbol{\sigma}(2)\ldots} (\boldsymbol{\mu}_{\mathrm{H}}(1) \otimes \{\boldsymbol{\sigma}(1)\}) \times (\boldsymbol{\mu}_{\mathrm{H}}(2) \otimes \{\boldsymbol{\sigma}(2)\}) \times \ldots \\ &\to \sum_{\boldsymbol{\sigma}(1),\boldsymbol{\sigma}(2),\ldots} m^{\boldsymbol{\lambda}}_{\boldsymbol{\sigma}(1)\boldsymbol{\sigma}(2)\ldots} (\boldsymbol{\mu}_{\mathrm{H}}(1) \otimes \{\boldsymbol{\sigma}(1)\}) \cdot (\boldsymbol{\mu}_{\mathrm{H}}(2) \otimes \{\boldsymbol{\sigma}(2)\}) \ldots \\ &= (\boldsymbol{\mu}_{\mathrm{H}}(1) + \boldsymbol{\mu}_{\mathrm{H}}(2) + \ldots) \otimes \{\boldsymbol{\lambda}\}, \end{aligned} \tag{6.5}$$

precisely as in (5.24).

It follows that the maximal embeddings in the unitary group are of the form

$$\mathrm{U}(d_{\mu_{\mathrm{H}}}) \supset \mathrm{H} \qquad \{1\} \to \boldsymbol{\mu}_{\mathrm{H}}. \tag{6.6}$$

Confining attention to the cases for which $\mathrm{H} = \mathrm{U}(n)$ this yields the maximal embeddings

$$\mathrm{U}(d_{\mu}) \supset \mathrm{U}(n) \qquad \{1\} \to e_{\mu} = \{\boldsymbol{\mu}\} = \chi^{\{\mu\}}_{\boldsymbol{\phi}} \tag{6.7}$$

with corresponding branching rules

$$\mathrm{U}(d_{\mu}) \supset \mathrm{U}(n) \qquad \{\boldsymbol{\lambda}\} \to e_{\mu} \otimes \{\boldsymbol{\lambda}\} = \{\boldsymbol{\mu}\} \otimes \{\boldsymbol{\lambda}\}, \tag{6.8}$$

where use has been made of (3.2) and (3.4).

It should be stressed that although (6.7) is a maximal embedding it is not, in general, a maximal subjoining. This is a consequence of the expansion

$$e_{\mu}(\boldsymbol{x}) = \sum_{\rho} K^{\rho}_{\mu} k_{\rho}(\boldsymbol{x}) = k_{\rho(1)}(\boldsymbol{x}) + k_{\rho(2)}(\boldsymbol{x}) + \ldots \tag{6.9}$$

where K^{ρ}_{μ} is a non-negative integer, which is an element of Kostka's matrix as discussed by Littlewood (1950, page 191) in terms of standard Young tableaux. These coefficients are nothing other than U(n) dominant weight multiplicities, as stressed more recently (King and Plunkett 1972, 1976) by virtue of the expansion of unitary group characters in accordance with (2.2):

$$\chi^{\{\mu\}}_{\boldsymbol{\phi}} = e_{\mu}(\boldsymbol{x}) = \sum_{\rho} K^{\rho}_{\mu} \sum_{\boldsymbol{m} \in C_{\rho}} \exp(\mathrm{i}\boldsymbol{m} \cdot \boldsymbol{\phi}) \tag{6.10}$$

so that

$$M^{\{\mu\}}_{\boldsymbol{m}} = M^{\{\mu\}}_{\rho} = K^{\rho}_{\mu} \qquad \text{for } \boldsymbol{m} \in C_{\rho}. \tag{6.11}$$

If the expansion (6.9) contains more than one term the maximal embedding (6.7) is a non-maximal subjoining as indicated by the existence of the chain

$$
\begin{aligned}
\mathrm{U}(d_{\mu}) \supset \mathrm{U}(c_{\rho(1)}) \times \mathrm{U}(c_{\rho(2)}) \times \ldots > \mathrm{U}(n) \times \mathrm{U}(n) \times \ldots \supset \mathrm{U}(n)& \\
\{1\} \to \{1\} \times \{0\} \times \ldots + \{0\} \times \{1\} \times \ldots + \ldots& \\
\to k_{\rho(1)} \times \{0\} \times \ldots + \{0\} \times k_{\rho(2)} \times \ldots + \ldots& \\
\to k_{\rho(1)} + k_{\rho(2)} + \ldots = e_{\mu}(\boldsymbol{x}),& \qquad (6.12)
\end{aligned}
$$

exactly as in (5.21). The corresponding branching rule yields (6.8) in the form

$$\{\boldsymbol{\mu}\} \otimes \{\boldsymbol{\lambda}\} = \Big(\sum_{\rho} K^{\rho}_{\mu} k_{\rho}(\boldsymbol{x})\Big) \otimes \{\boldsymbol{\lambda}\}. \qquad (6.13)$$

Just as the only subjoinings defined by (4.1) which may be maximal are those of the form (3.13), corresponding to the identity $S_r = k_r$, so the only embeddings defined by (6.7) which may be maximal subjoinings are those for which $\boldsymbol{\mu} = \{1^r\}$ corresponding to the identity $e_{1^r} = k_{1^r}$. Such an example is provided by (5.14).

7. Branching rule evaluation

It has been tacitly assumed in all the previous sections that tables of plethysms of the form $\{\boldsymbol{\mu}\} \otimes \{\boldsymbol{\lambda}\}$ are available, as indeed they are in the work of Ibrahim (1950), Butler and Wybourne (1971) and Vanagas (1971), for example. This together with the algebra of plethysms given by Littlewood (1950, page 290) leads to the validity of all the results of the form (3.18), (4.6), (4.10), (5.12) and (5.17) given here. It should be pointed out that quite apart from the use of recurrence relations and the like to obtain such tables, the most straightforward way of defining a plethysm is as a mapping between weight spaces (King and Plunkett 1972) corresponding to a substitutional operation as stressed by Read (1968) and Thomas (1976). This definition allows for the evaluation of plethysms of any sort of symmetric functions by a straightforward enumerative procedure.

In general the subjoining of U(n) to U(N) is defined by

$$
\begin{aligned}
\mathrm{U}(N) > \mathrm{U}(n) \qquad & \{1\} = \exp(\mathrm{i}\psi_1) + \exp(\mathrm{i}\psi_2) + \ldots + \exp(\mathrm{i}\psi_N) = \sum_{j=1}^{N} y_j \\
& \to q(\boldsymbol{x}) = \sum_{\mu} B^{\{1\}}_{\{\mu\}} \{\boldsymbol{\mu}\} = \sum_{\mu} B^{\{1\}}_{\{\mu\}} e_{\mu}(\boldsymbol{x}) \\
& = \sum_{\mu} B^{\{1\}}_{\{\mu\}}\, \mathrm{e}_{\mu}(\exp(\mathrm{i}\phi_1), \exp(\mathrm{i}\phi_2), \ldots, \exp(\mathrm{i}\phi_n)) \qquad (7.1)
\end{aligned}
$$

where, in addition to using (3.3), it has been convenient to introduce

$$y_j = \exp(\mathrm{i}\psi_j) \qquad \text{for } j = 1, 2, \ldots, N. \qquad (7.2)$$

Here N is the number of terms in the expansion of the right-hand side to give the particular symmetric function $q(\boldsymbol{x})$, so that

$$N = q(1, 1, \ldots 1) = \sum_{\mu} B^{\{1\}}_{\{\mu\}} d_{\mu}. \qquad (7.3)$$

The corresponding branching rule is then

$$\mathrm{U}(N) > \mathrm{U}(n) \qquad \{\boldsymbol{\lambda}\} \to q(\boldsymbol{x}) \otimes \{\boldsymbol{\lambda}\}, \tag{7.4}$$

which may be evaluated by noting that, just as in (7.1),

$$\{1\} = e_1(\boldsymbol{y}) = q(\boldsymbol{x}), \tag{7.5}$$

so in (7.4)

$$\{\boldsymbol{\lambda}\} = e_{\boldsymbol{\lambda}}(\boldsymbol{y}) = q(\boldsymbol{x}) \otimes \{\boldsymbol{\lambda}\}. \tag{7.6}$$

For example, if

$$\{1\} = e_1(\boldsymbol{y}) = y_1 + y_2 + \ldots + y_n = \mathrm{S}_3(\boldsymbol{x}) = x_1^3 + x_2^3 + \ldots + x_n^3 \tag{7.7}$$

as in (3.17), then

$$\begin{aligned}\{2\} = e_2(\boldsymbol{y}) &= y_1^2 + y_1y_2 + y_1y_3 + \ldots + y_1y_n + y_2^2 + y_2y_3 + \ldots + y_2y_n + \ldots + y_n^2 \\ &= x_1^6 + x_1^3x_2^3 + x_1^3x_3^3 + \ldots + x_1^3x_n^3 + x_2^6 + x_2^3x_3^3 + \ldots x_2^3x_n^3 + \ldots + x_n^6 \\ &= k_6(\boldsymbol{x}) + k_{3^2}(\boldsymbol{x}) \\ &= e_6(\boldsymbol{x}) - e_{51}(\boldsymbol{x}) + e_{41^2}(\boldsymbol{x}) + e_{3^2}(\boldsymbol{x}) - e_{321}(\boldsymbol{x}) + e_{2^3}(\boldsymbol{x}),\end{aligned} \tag{7.8}$$

where the last line may be seen to be valid by consulting the weight multiplicity table:

	k_6	k_{51}	k_{42}	k_{41^2}	k_{3^2}	k_{321}	k_{31^3}	k_{2^3}	$k_{2^21^2}$	k_{21^4}	k_{1^6}
e_6	1	1	1	1	1	1	1	1	1	1	1
$-e_{51}$		−1	−1	−2	−1	−2	−3	−2	−3	−4	−5
e_{41^2}				1	0	1	3	1	3	6	10
e_{3^2}					1	1	1	1	2	3	5
$-e_{321}$						−1	−2	−2	−4	−8	−16
e_{2^3}								1	1	2	5
$\mathrm{S}_3 \otimes e_2$	1	0	0	0	1	0	0	0	0	0	0

obtained from a more complete tabulation (King and Plunkett 1976). Alternatively, use could be made more directly of the matrix L in (5.6) which is the inverse of the weight multiplicity matrix K of (6.10). This result (7.8) confirms (3.18).

Similarly, if

$$\begin{aligned}\{1\} = e_1(\boldsymbol{y}) &= y_1 + y_2 + \ldots + y_{n(n-1)} = k_{21}(\boldsymbol{x}) \\ &= x_1^2x_2 + x_1^2x_3 + \ldots + x_1^2x_n + x_2^2x_1 + x_2^2x_3 + \ldots\end{aligned} \tag{7.9}$$

as in (5.8), then

$$\begin{aligned}\{2\} = e_2(\boldsymbol{y}) &= y_1^2 + y_1y_2 + \ldots y_1y_{n(n-1)} + y_2^2 + y_2y_3 + \ldots \\ &= x_1^4x_2^2 + x_1^4x_2x_3 + \ldots + x_1^2x_2x_{n-1}x_n^2 + x_1^4x_3^2 + x_1^4x_3x_4 + \ldots.\end{aligned} \tag{7.10}$$

In order to evaluate this expression it is only necessary to establish in how many distinct ways the leading term $x_1^{\rho_1}x_2^{\rho_2}\ldots$ of $k_{\boldsymbol{\rho}}(\boldsymbol{x})$ may be formed for each partition $\boldsymbol{\rho}$ of 6. These leading terms arise in the following way:

$$
\begin{aligned}
&y_1^2 = x_1^4x_2^2 && (2\ 1\ 0\ \ldots)(2\ 1\ 0\ \ldots)\to(4\ 2\ 0\ \ldots)\\
&y_1y_2 = x_1^4x_2x_3 && (2\ 1\ 0\ \ldots)(2\ 0\ 1\ \ldots)\to(4\ 1\ 1\ \ldots)\\
&y_1y_n = x_1^3x_2^3 && (2\ 1\ 0\ \ldots)(1\ 2\ 0\ \ldots)\to(3\ 3\ 0\ \ldots)\\
&y_2y_n = x_1^3x_2x_3 && (2\ 0\ 1\ \ldots)(1\ 2\ 0\ \ldots)\to(3\ 2\ 1\ \ldots)\\
&y_1y_{2n} = y_2y_{n+1} = y_ny_{2n-1} = x_1^2x_2^2x_3^2\\
&(2\ 1\ 0\ 0\ \ldots)(0\ 1\ 2\ 0\ \ldots)\to(2\ 2\ 2\ 0\ \ldots)\\
&(2\ 0\ 1\ 0\ \ldots)(0\ 2\ 1\ 0\ \ldots)\to(2\ 2\ 2\ 0\ \ldots)\\
&(1\ 2\ 0\ 0\ \ldots)(1\ 0\ 2\ 0\ \ldots)\to(2\ 2\ 2\ 0\ \ldots)\\
&y_2y_{n+2} = y_3y_{n+1} = x_1^2x_2^2x_3x_4\\
&(2\ 0\ 1\ 0\ \ldots)(0\ 2\ 0\ 1\ \ldots)\to(2\ 2\ 1\ 1\ \ldots)\\
&(2\ 0\ 0\ 1\ \ldots)(0\ 2\ 1\ 0\ \ldots)\to(2\ 2\ 1\ 1\ \ldots).
\end{aligned}
\qquad (7.11)
$$

This is sufficient to show that:

$$
\begin{aligned}
\{2\} = e_2(\boldsymbol{y}) &= k_{42}(\boldsymbol{x}) + k_{41^2}(\boldsymbol{x}) + k_{3^2}(\boldsymbol{x}) + k_{321}(\boldsymbol{x}) + 3k_{2^3}(\boldsymbol{x}) + 2k_{2^21^2}(\boldsymbol{x})\\
&= e_{42}(\boldsymbol{x}) - e_{321}(\boldsymbol{x}) - e_{31^3}(\boldsymbol{x}) + 2e_{2^3}(\boldsymbol{x}) + e_{2^21^2}(\boldsymbol{x}) - e_{21^4}(\boldsymbol{x}) + 3e_{1^6}(\boldsymbol{x}),
\end{aligned}
\qquad (7.12)
$$

in agreement with (5.12), where once again recourse has been made to weight multiplicity tables (King and Plunkett 1976) which yield:

	k_{42}	k_{41^2}	k_{3^2}	k_{321}	k_{31^3}	k_{2^3}	$k_{2^21^2}$	k_{21^4}	k_{1^6}
e_{42}	1	1	1	2	3	3	4	6	9
$-e_{321}$				−1	−2	−2	−4	−8	−16
$-e_{31^3}$					−1	0	−1	−4	−10
$2e_{2^3}$						2	2	4	10
$e_{2^21^2}$							1	3	9
$-e_{21^4}$								−1	−5
$3e_{1^6}$									3
$k_{21}\otimes e_2$	1	1	1	1	0	3	2	0	0

This last example illustrates that the method is both powerful and straightforward when maximum use is made of the Weyl symmetry groups of $U(N)$ and $U(n)$ to limit the required projections to those of the type (7.11) onto the highest weights $\boldsymbol{\rho}$ of each set C_ρ. This has previously been used in the evaluation of S-function plethysms (King and Plunkett 1972) which depend, as do the calculations here, essentially upon the evaluation of the coefficients in

$$k_\mu \otimes k_\lambda = \sum_\nu P^\nu_{\mu\lambda}k_\nu. \qquad (7.13)$$

Thus only the first term of (7.11) contributes to $k_{21}\otimes k_2$, whilst all the others contribute to $k_{21}\otimes k_{1^2} = k_{21}\otimes e_{1^2}$. Hence, as in (7.12),

$$
\begin{aligned}
\{1^2\} = e_{1^2}(\boldsymbol{y}) &= k_{41^2}(\boldsymbol{x}) + k_{3^2}(\boldsymbol{x}) + k_{321}(\boldsymbol{x}) + 3k_{2^3}(\boldsymbol{x}) + 2k_{21}(\boldsymbol{x})\\
&= e_{41^2}(\boldsymbol{x}) + e_{3^2}(\boldsymbol{x}) - e_{321}(\boldsymbol{x}) - 2e_{31^3}(\boldsymbol{x}) + 3e_{2^3}(\boldsymbol{x}) + e_{21^4}(\boldsymbol{x}) + e_{1^6}(\boldsymbol{x})
\end{aligned}
$$

by virtue of the table

	k_{41^2}	k_{3^2}	k_{321}	k_{31^3}	k_{2^3}	$k_{2^21^2}$	k_{21^4}	k_{1^6}
e_{41^2}	1	0	1	3	1	3	6	10
e_{3^2}		1	1	1	1	2	3	5
$-e_{321}$			−1	−2	−2	−4	−8	−16
$-2e_{31^3}$				−2	0	−2	−8	−20
$3e_{2^3}$					3	3	6	15
e_{21^4}							1	5
e_{1^6}								1
$k_{21} \otimes e_{1^2}$	1	1	1	0	3	2	0	0

8. Conclusion

From the remarks at the beginning of § 5 it is easy to generalise the maximal subjoinings of U(n) to U(N) specified by (5.3), to the case of subjoinings of any compact Lie group H to U(N). The corresponding specification is

$$\mathrm{U}(N) > \mathrm{H} \qquad \{1\} \to w_{\rho_{\mathrm{H}}} = \sum_{m \in C_{\rho_{\mathrm{H}}}} \exp(\mathrm{i}\boldsymbol{m} \cdot \boldsymbol{\phi}) \tag{8.1}$$

where $\boldsymbol{\rho}_{\mathrm{H}}$ is an H-dominant weight of H (King and Al-Qubanchi 1980)

$$C_{\rho_{\mathrm{H}}} = \{\boldsymbol{m} : \boldsymbol{m} = T\boldsymbol{\rho}_{\mathrm{H}},\ T \in \mathrm{W}_{\mathrm{H}}\}$$

and

$$N = c_{\rho_{\mathrm{H}}} = |C_{\rho_{\mathrm{H}}}|.$$

The corresponding branching rules take the form:

$$\mathrm{U}(c_{\rho_{\mathrm{H}}}) > \mathrm{H} \qquad \{\boldsymbol{\lambda}\} \to w_{\rho_{\mathrm{H}}} \otimes \{\boldsymbol{\lambda}\} \tag{8.2}$$

with this plethysm defined by the mapping between weight spaces fixed, up to equivalence under transformations of the type (2.9), by (8.1).

It is thus a straightforward matter to determine subjoinings of the type (8.1) by studying the weight multiplicity tables of both the classical (King and Plunkett 1976) and exceptional (King and Al-Qubanchi 1978, 1980) Lie groups. To be precise, if the dominant weight multiplicities of these groups are denoted by $M^{\mu_{\mathrm{H}}}_{\rho_{\mathrm{H}}}$, where $\boldsymbol{\rho}_{\mathrm{H}}$ specifies the dominant weight and $\boldsymbol{\mu}_{\mathrm{H}}$ the irreducible representation in which this weight has multiplicity $M^{\mu_{\mathrm{H}}}_{\rho_{\mathrm{H}}}$, then the triangular nature of the weight multiplicity matrix, which has each of its diagonal elements equal to 1, is such that it is easy to invert. This yields a matrix whose elements define the expansion analogous to (5.5):

$$w_{\rho_{\mathrm{H}}} = \sum_{\mu_{\mathrm{H}}} L^{\mu_{\mathrm{H}}}_{\rho_{\mathrm{H}}} \boldsymbol{\mu}_{\mathrm{H}}. \tag{8.3}$$

This relation allows (8.1) to be written in the required form, namely

$$\mathrm{U}(N) > \mathrm{H} \qquad \{1\} \to w_{\rho_{\mathrm{H}}} = \sum_{\mu_{\mathrm{H}}} L^{\mu_{\mathrm{H}}}_{\rho_{\mathrm{H}}} \boldsymbol{\mu}_{\mathrm{H}} \tag{8.4}$$

with

$$N=\sum_{\mu_H} L^{\mu_H}_{\rho_H} d_{\mu_H} = c_{\rho_H}. \tag{8.5}$$

This specification serves to define

$$Q = Q_{U(N)H} = Q(w_{\rho_H})$$

and the corresponding branching rule takes the form

$$U(N) > H \qquad \{\boldsymbol{\lambda}\} \to w_{\rho_H} \otimes \{\boldsymbol{\lambda}\} = \Big(\sum_{\mu_H} L^{\mu_H}_{\rho_H} \boldsymbol{\mu}_H\Big) \otimes \{\boldsymbol{\lambda}\}. \tag{8.6}$$

In general the subjoinings (8.1) are not maximal in the sense that there may exist a chain of the form

$$U(N) \supset G > H \qquad \{1\} \to \boldsymbol{\omega}_G \to w_{\rho_H} \tag{8.7}$$

where $\boldsymbol{\omega}_G$ is the defining representation of a group G of dimension $d_{\omega_G} = N = c_{\rho_H}$. In the case of a semi-simple compact Lie group H some such group G almost invariably exists for each $\boldsymbol{\rho}_H$. This group G may be anything from SU(N) to O(N), Sp(N) or even H itself. The problem is to determine G such that the second subjoining of (8.7)

$$G > H \qquad \boldsymbol{\omega}_G \to w_{\rho_H} \tag{8.8}$$

is maximal.

Furthermore, for certain groups G the defining representation $\boldsymbol{\omega}_G$ contains a second G-dominant weight in addition to $\boldsymbol{\omega}_G$ itself. This is the null vector $\mathbf{0}$ which has non-vanishing multiplicity

$$n_G = M^{\omega_G}_{\mathbf{0}} \tag{8.9}$$

in the cases for which G is SO($2k+1$), G_2, F_4 or E_8. There then exist chains of the form

$$U(N) > G > H \qquad \{1\} \to \boldsymbol{\omega}_G - n_G\mathbf{0} \to w_{\rho_H} \tag{8.10}$$

with the second subjoining, which is maximal, defined by

$$G > H \qquad \boldsymbol{\omega}_G \to w_{\rho_H} + n_G\mathbf{0}. \tag{8.11}$$

It is hoped to take up these points in a forthcoming paper.

At this stage it suffices to claim that the present paper sheds further light on the concept of subjoining introduced by Patera and Sharp (1980). For example, the special case of (3.13) with $n=2$ and the imposition of the unimodular condition gives

$$\begin{aligned} SU(2) > SU(2) \qquad & \{1\} \to S_r = \exp(ir\phi_1) + \exp(ir\phi_2) = \exp(ir\phi) + \exp(-ir\phi) \\ &= \{r\} - \{r-1, 1\} \\ &= \{r\} - \{r-2\}, \end{aligned} \tag{8.12}$$

in accordance with the result (3.2) of Patera *et al* (1980), whilst (3.13), (4.1), (5.6) and (6.7) provide a wealth of generalisations. For other groups (8.7) and (8.10) provide the key to the determination of all maximal subjoinings. The only remaining task is the identification of G. This has been done in some cases by Patera *et al* (1980) whose illustrations of maximal subjoinings can all be shown to be of the form (8.8) or (8.11).

References

Butler P H and Wybourne B G 1971 *Atomic Data* **3** 133–51

Dynkin E B 1957 *Am. Math. Soc. Transl. Ser. 2* **6** 111–244

Foulkes H O 1951 *J. Lond. Math. Soc.* **26** 132–9

Frobenius G 1900 *Sitz. Ber. Preuss. Akad. (Berlin)* 516–34

Hall P 1959 *Proc. 4th Canadian Math. Congr., Banff* 1957 (Toronto: University of Toronto Press) pp 147–59

Ibrahim E M 1950 *Tables for the Plethysm of S-functions* (Royal Society, London: Depository of Unpublished Tables)

King R C 1980 *J. Phys. A: Math. Gen.* to be published. *Maths Faculty Preprint* 41 (University of Southampton)

King R C and Al-Qubanchi A H A 1978 *J. Phys. A: Math. Gen.* **11** 1491–9

—— 1980 *J. Phys. A: Math. Gen.* to be published. *Maths Faculty Preprint* 38 (University of Southampton)

King R C and Plunkett S P O 1972 *J. Physique* **33** 145–53

—— 1976 *J. Phys. A: Math. Gen.* **9** 863–87

Littlewood D E 1950 *The Theory of Group Characters* 2nd edn (London: Oxford University Press)

—— 1951 *Proc. R. Soc.* A **209** 333–52

Murnaghan F D 1938 *Theory of Group Representations* (Baltimore: John Hopkins Press)

Patera J and Sharp R T 1980 *J. Phys. A: Math. Gen.* **13** 397–416

Patera J, Sharp R T and Slansky R 1980 *Los Alamos Scientific Laboratory Report LA-UR-80-89* (submitted for publication)

Plunkett S P O 1972 *Can. J. Math.* **24** 541–52

Read R C 1968 *Can. J. Math.* **20** 808–41

Stanley R P 1971 *Stud. Appl. Math.* **50** 167–88

Thomas G P 1976 *Can. J. Math.* **28** 879–84

Vanagas V 1971 *Algebraic Methods in Nuclear Theory* (Vilnius: Mintis)